GIESEKER MODULI SPACE OF BUNDLES ON $\mathbb{P}^2$ AS A NAKAJIMA QUIVER VARIETY

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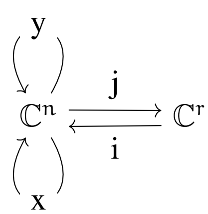
1. Introduction

We consider the moduli space of rank r coherent torsion-free sheaves E on $\mathbb{P}^2$ with fixed trivialization on the line ℓ_∞, i.e. $E_{\ell_\infty} \cong \mathcal{O}^{\oplus r}$ (this implies $c_1(E) = 0$ as $H^2(\mathbb{P}^2, \mathbb{Z})$ is generated by ℓ_∞) and $c_2(E) = n$, up to isomorphisms. This moduli space will be denoted by $\mathcal{M}_{r,n}$. Our goal is to explain an isomorphism of $\mathcal{M}_{r,n}$ with the Nakajima quiver variety



$$\left\{ [x, y, i, j] \in (\mathrm{End}(\mathbb{C}^n)^{\oplus 2} \times \mathrm{Hom}_{rn} \times \mathrm{Hom}_{nr}) \left| \begin{array}{l} [x, y] + ij = 0; \\ \text{Stability: there is no subspace } S \subset \mathbb{C}^n, \\ \text{such that } x(S), y(S) \subset S \text{ and } im(i) \subset S \end{array} \right. \right\} \Big/ GL_n(\mathbb{C}),$$

where $\mathrm{Hom}_{rn} = \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^n)$, $\mathrm{Hom}_{nr} = \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^r)$ and $g \cdot (x, y, i, j) = (gxg^{-1}, gyg^{-1}, gi, jg^{-1})$.

These notes are mostly based on lectures [Nak99] and chapter 2.3 of the book [OSS11].

2. Beilinson Spectral Sequence and Monad Description

First, we describe a construction which allows to study torsion-free sheaves using linear algebra, namely, the sheaf is presented as a monad, which is a complex presented below, with $\ker(a) = \mathrm{coker}(b) = 0$ and $E \cong \ker(b)/\mathrm{im}(a)$

$$0 \to A \xrightarrow{a} B \xrightarrow{b} C \to 0.$$

2.1. **Resolutions of Coherent Sheaves on $\mathbb{P}^n$.** Let us remind the construction of Beilinson. We take the following resolution of the diagonal $\triangle \subset \mathbb{P}^n \times \mathbb{P}^n$. Define Q from the SES

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-1) \to \mathcal{O}_{\mathbb{P}^n}^{\oplus n+1} \to Q \to 0$$

Notation. For coherent sheaves F, G on $\mathbb{P}^n$ we set $F \boxtimes G := pr_1^* F \otimes pr_2^* G$ as sheaves on $\mathbb{P}^n \times \mathbb{P}^n$, where

$$\begin{array}{ccc} \mathbb{P}^n \times \mathbb{P}^n & \xrightarrow{pr_1} & \mathbb{P}^n \\ \downarrow{\scriptstyle pr_2} & & \\ \mathbb{P}^n & & \end{array}$$

$$\mathcal{O}_{\mathbb{P}^n}(1) \boxtimes Q := \mathcal{H}om(pr_1^*(\mathcal{O}_{\mathbb{P}^n}(-1)), pr_2^*(Q)).$$

Next, define the section s of this bundle, which over a point $(x, y) \in \mathbb{P}^n \times \mathbb{P}^n$, corresponding to the lines $\ell, v \in \mathbb{C}^{n+1}$, is $s_{(x,y)} \in \mathrm{Hom}_{\mathbb{C}}(\mathcal{O}_{\mathbb{P}^n}(-1)_x, Q_y)$, $\ell \mapsto [\ell]$ - the class of ℓ in the factor space $\mathbb{C}^{n+1}/\mathbb{C}v = Q(y)$. Clearly, the diagonal is the kernel of this map, i.e. $\triangle = s^{-1}(0)$. We produce the other terms the same way as for the Koszul resolution:

$$0 \to \Lambda^n(\mathcal{O}_{\mathbb{P}^n}(-1) \boxtimes Q^\vee) \to \cdots \to \Lambda^2(\mathcal{O}_{\mathbb{P}^n}(-1) \boxtimes Q^\vee) \to$$

$$\to \mathcal{O}_{\mathbb{P}^n}(-1) \boxtimes Q^\vee \xrightarrow{s} \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n} \to \mathcal{O}_\triangle \to 0$$

Now we tensor this sequence with $pr_2^* E$ to obtain

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-n) \boxtimes (E \otimes \Omega^n_{\mathbb{P}^n}(n)) \to \cdots \to \mathcal{O}_{\mathbb{P}^n}(-2) \boxtimes (E \otimes \Omega^2_{\mathbb{P}^n}(2)) \to \mathcal{O}_{\mathbb{P}^n}(-1) \boxtimes (E \otimes \Omega^1_{\mathbb{P}^n}(1)) \to \mathcal{O}_{\mathbb{P}^n} \boxtimes E \to 0.$$

Fix notation: $C_{-i} := \mathcal{O}_{\mathbb{P}^n}(-i) \boxtimes (E \otimes \Omega^i_{\mathbb{P}^n}(i))$, $C^\bullet$ denotes the complex above.

2.2. **Beilinson Spectral Sequence.** Construct an injective (i.e. Cech with an appropriate cover of $\mathbb{P}^n \times \mathbb{P}^n$) resolution of each term of $C^\bullet$ to come up with a double complex $I^{\bullet\bullet}$.

$$\begin{array}{ccccccccc} \cdots & & & & & & & & \\ I^{(-n,1)} & \longrightarrow & \cdots & \longrightarrow & I^{(-2,1)} & \longrightarrow & I^{(-1,1)} & \longrightarrow & I^{(0,1)} \\ \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ I^{(-n,0)} & \longrightarrow & \cdots & \longrightarrow & I^{(-2,0)} & \longrightarrow & I^{(-1,0)} & \longrightarrow & I^{(0,0)} \end{array}$$

Our next goal is to compute cohomology of the total complex $pr_{1_*}(I^{\bullet\bullet})$ using (separately) two spectral sequences $'E$ and $''E$. The E_2-terms are

$$'E_2^{pq} = H^p(R^q pr_{1_*}(C^\bullet))$$

$$''E_2^{pq} = R^p pr_{1_*}(H^q(C^\bullet))$$

Consider the following obvious identity: for a coherent sheaf E on $\mathbb{P}^2$

$$pr_{1*}(pr_2^* E \otimes \mathcal{O}_\triangle) = E.$$

This helps us to figure out that

$$''E_2^{pq} = R^p pr_{1_*}(H^q(C^\bullet)) = \begin{cases} E & (p,q) = (0,0) \\ 0, & \text{otherwise} \end{cases}.$$

2.3. **Application to Coherent Sheaves on $\mathbb{P}^2$.** We will need the following technical results, the proofs of which are explained in Appendix A.

Theorem 1. Let G, F be coherent sheaves on a compact variety X, moreover, F is locally free. Then $Rpr_{1*}(F \boxtimes G) \cong F \otimes H^\bullet(G)$.

Theorem 2. Let E be a torsion-free coherent sheaf on $\mathbb{P}^2$, locally free on l_∞, then

$$\begin{cases} H^q(\mathbb{P}^2, E(-p)) = 0, & p = 1,2, q = 0,2 \\ H^q(\mathbb{P}^2, E(-1) \otimes Q^\vee) = 0, & q = 0,2 \end{cases}.$$

Notice that $\Lambda^2 Q^\vee \cong \mathcal{O}_{\mathbb{P}^2}(-1)$, therefore, $E(-1) \otimes \Lambda^2 Q^\vee \cong E(-2)$. So if we take $E(-1)$ instead of E, the first page of the Beilinson spectral sequence provides us with

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-2) \otimes H^q(\mathbb{P}^2, E(-2)) \xrightarrow{a_q'} \mathcal{O}_{\mathbb{P}^2}(-1) \otimes H^q(\mathbb{P}^2, E(-1) \otimes Q^\vee) \xrightarrow{b_q'} \mathcal{O}_{\mathbb{P}^2} \otimes H^q(\mathbb{P}^2, E(-1)) \to 0,$$

which, according to Theorem 2, is nonzero if and only if $q = 1$. It follows that the spectral sequence $'E$ also degenerates on the second page. As $\bigoplus_{p+q=0} {}'E_2^{p,q} = \bigoplus_{p+q=0} {}''E_2^{p,q} = E(-1)$ and $\bigoplus_{p+q\neq 0} {}'E_2^{p,q} = \bigoplus_{p+q\neq 0} {}''E_2^{p,q} = 0$, we see that $\ker a = \operatorname{coker} b = 0, E(-1) \cong \ker b_1' / \operatorname{im} a_1'$. We tensor the monad for $E(-1)$ with $\mathcal{O}_{\mathbb{P}^2}(1)$ to obtain the monad for E.

The next step is to use the monad description of E for identification with the one provided by Nakajima quiver variety. From the first page of Beilinson spectral sequence $'E$, we have the sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-1) \otimes V \xrightarrow{a} \mathcal{O}_{\mathbb{P}^2} \otimes \tilde{W} \xrightarrow{b} \mathcal{O}_{\mathbb{P}^2}(1) \otimes V' \to 0,$$

where $\ker a = \operatorname{coker} b = 0$ and $E \cong \ker b / \operatorname{im} a$, $V := H^1(\mathbb{P}^2, E(-2))$, $V' := H^1(\mathbb{P}^2, E(-1))$ and $\tilde{W} := H^1(\mathbb{P}^2, E(-1) \otimes Q)$.

Lemma. $\dim V = \dim V' = c_2(E)$, $\dim \tilde{W} = 2c_2(E) + rk(E)$.

Proof. We demonstrate the calculation of $\dim V$, the other two equations are derived analogously. Use the splitting principle: $E = E_1 \oplus E_2 \oplus \cdots \oplus E_r$, where each E_i is a line bundle. Then $c(E) = \prod_{i=1}^{r}(1 + c_1(E_i))$, $E(-2) = E_1 \otimes \mathcal{O}(-2) \oplus E_2 \otimes \mathcal{O}(-2) \oplus \cdots \oplus E_r \otimes \mathcal{O}(-2)$. The following formula is due to Hirzebruch:

$$\chi(E) = Ch(E)Td(T_X)_n \text{ (*)},$$

where $Ch(E) = \sum_{i=1}^{r} e^{\alpha_i}$, $Td(E) = \prod_{i=1}^{r} \frac{\alpha_i}{1-e^{-\alpha_i}}$, $\alpha_i = c_1(E_i)$ and the subscript n corresponds to the component of degree n (each α_i has degree 1). From the Euler exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{P}^2}^{\oplus 3}(1) \to T_{\mathbb{P}^2} \to 0$$

$$c(T_{\mathbb{P}^2}) = 1 + 3H + 3H^2,$$

where H is the class of hyperplane. From the formula for $Td(E)$ it is not hard to see that

$$Td_0(E) = 1,$$

$$Td_1(E) = \frac{c_1(E)}{2},$$

$$Td_2(E) = \frac{c_1^2(E) + c_2(E)}{12},$$

so $Td_1(T_{\mathbb{P}^2}) = \frac{3H}{2}$, $Td_2(T_{\mathbb{P}^2}) = H^2$.

$$Ch_0(E) = rk(E),$$

$$Ch_1(E) = c_1(E),$$

$$Ch_2(E) = \frac{c_1^2(E) - 2c_2(E)}{2},$$

$$Ch_1(E(-2)) = c_1(E(-2)) = \sum_{i=1}^{r}(\alpha_i - 2) = \sum_{i=1}^{r} \alpha_i - 2r = c_1(E) - 2r = -2r$$

$$c_2(E(-2)) = \text{coefficient of } H^2 \text{ in } \prod_{i=1}^{r}((\alpha_i - 2)H) = n + 4\binom{r}{2}, Ch_2(E(-2)) = n + 2r$$

Applying the formula $(*)$ and using Theorem 2, we get

$$-\dim V = -n + 2r - \frac{3}{2} \cdot 2r + r = -n.$$

□

We now take $a \in \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^2}(-1) \otimes V, \mathcal{O}_{\mathbb{P}^2} \otimes \tilde{W}) \cong \mathcal{O}_{\mathbb{P}^2}(1) \otimes \mathrm{Hom}(V, \tilde{W})$. In coordinates $[z_0 : z_1 : z_2]$ on $\mathbb{P}^2$, $a = z_0 a_0 + z_1 a_1 + z_2 a_2$, where $a_i \in \mathrm{Hom}(V, \tilde{W})$, similarly, $b = z_0 b_0 + z_1 b_1 + z_2 b_2$, $b_i \in \mathrm{Hom}(\tilde{W}, V')$. Recall that $ba = 0$, which gives us six equations:

$$\begin{cases} b_0 a_0 = 0, & b_0 a_1 + b_1 a_0 = 0, \\ b_1 a_1 = 0, & b_1 a_2 + b_2 a_1 = 0, \\ b_2 a_2 = 0, & b_0 a_2 + b_2 a_0 = 0. \end{cases}$$

Next, we restrict the monad to ℓ_∞:

$$0 \to \mathcal{O}_{\ell_\infty}(-1) \otimes V \xrightarrow{a|_{\ell_\infty}} \mathcal{O}_{\ell_\infty} \otimes \tilde{W} \xrightarrow{b|_{\ell_\infty}} \mathcal{O}_{\ell_\infty}(1) \otimes V' \to 0,$$

$$\begin{cases} a|_{\ell_\infty} = z_1 a_1 + z_2 a_2 \\ b|_{\ell_\infty} = z_1 b_1 + z_2 b_2 \end{cases}.$$

Proposition. Consider the SES $0 \to \mathcal{O}_{\ell_\infty}(-1) \otimes V \xrightarrow{a|_{\ell_\infty}} \ker b_{\ell_\infty} \to E|_{\ell_\infty} \to 0$. Then $H^0(\ell_\infty, \ker b_{\ell_\infty}) \simeq H^0(\ell_\infty, E|_{\ell_\infty}), H^1(\ell_\infty, \ker b|_{\ell_\infty}) = 0$.

Proof. From the long exact sequence

$$0 \to H^0(\ell_\infty, \mathcal{O}_{\ell_\infty}(-1)) \otimes V \to H^0(\ell_\infty, \ker b|_{\ell_\infty}) \to H^0(\ell_\infty, E|_{\ell_\infty})$$

$$\to H^1(\ell_\infty, \mathcal{O}_{\ell_\infty}(-1)) \otimes V \to H^1(\ell_\infty, \ker b|_{\ell_\infty}) \to H^1(\ell_\infty, E|_{\ell_\infty}) \to 0,$$

using that

$$H^0(\ell_\infty, \mathcal{O}_{\ell_\infty}(-1)) = H^1(\ell_\infty, \mathcal{O}_{\ell_\infty}(-1)) = 0,$$

we see

$$\begin{cases} H^0(\ell_\infty, \ker b|_{\ell_\infty}) \simeq H^0(\ell_\infty, E|_{\ell_\infty}) \\ H^1(\ell_\infty, \ker b|_{\ell_\infty}) \simeq H^1(\ell_\infty, E|_{\ell_\infty}). \end{cases}$$

Furthermore, $E|_{\ell_\infty} \simeq \mathcal{O}^{\oplus r}$, which implies $H^1(\ell_\infty, \ker b|_{\ell_\infty}) \simeq H^1(\ell_\infty, E|_{\ell_\infty}) = 0$ and $H^0(\ell_\infty, \ker b|_{\ell_\infty}) \simeq H^0(\ell_\infty, E|_{\ell_\infty})$ is a vector space of dimension r. □

Corollary. There exists an exact sequence

$$0 \to H^0(\ell_\infty, \ker b|_{\ell_\infty}) \to \tilde{W} \to V' \oplus V' \to 0.$$

Proof. From the SES $0 \to \ker b|_{\ell_\infty} \to \mathcal{O}_{\ell_\infty} \otimes \tilde{W} \to \mathcal{O}_{\ell_\infty}(1) \otimes V' \to 0$, obtain long exact sequence of cohomology:

$$0 \to H^0(\ell_\infty, \ker\ b|_{\ell_\infty}) \to H^0(\ell_\infty, \mathcal{O}_{\ell_\infty}) \otimes \tilde{W} \to H^0(\ell_\infty, \mathcal{O}_{\ell_\infty}(1) \otimes V')$$

$$\to H^1(\ell_\infty, \ker\ b|_{\ell_\infty}) \to H^1(\ell_\infty, \mathcal{O}_{\ell_\infty}) \otimes \tilde{W} \to H^1(\ell_\infty, \mathcal{O}_{\ell_\infty}(1) \otimes V') \to 0.$$

As $H^1(\ell_\infty, \ker b|_{\ell_\infty}) = 0$, $H^0(\ell_\infty, \mathcal{O}_{\ell_\infty}) \cong \mathbb{C}$ and $H^0(\ell_\infty, \mathcal{O}_{\ell_\infty}(1)) \cong \mathbb{C}z_1 \oplus \mathbb{C}z_2$, the assertion holds. □

Set $W := H^0(\ell_\infty, \ker b|_{\ell_\infty})$. The corollary, in particular, shows that $\dim W = 2r + n - 2r = n$.

Next, consider the dual to our monad, restricted to ℓ_∞, namely,

$$0 \to \mathcal{O}_{\ell_\infty}(-1) \otimes V'^* \xrightarrow{b^t|_{\ell_\infty}} \mathcal{O}_{\ell_\infty} \otimes \tilde{W}^* \xrightarrow{a^t|_{\ell_\infty}} \mathcal{O}_{\ell_\infty}(1) \otimes V^* \to 0.$$

Performing manipulations similar to the above, we come up with the SES

$$0 \to H^0(\ker a^t|_{\ell_\infty}) \to \tilde{W}^* \xrightarrow{(a_1^t, a_2^t)} V^* \oplus V^* \to 0,$$

so $(a_1, a_2) : V \oplus V \to \tilde{W}$ is injective. Also, $0 = \operatorname{im} a_1 \cap \ker b_2$, thus, $b_2 a_1 = -b_1 a_2 : V \simeq V'$ are isomorphisms (they are injective, the dimensions of V and V' are equal).

The six equations derived from $ba = 0$ enable us to give the presentation $a_0 = \begin{pmatrix} x \\ y \\ j \end{pmatrix}$

$a_1 = \begin{pmatrix} id_V \\ 0 \\ 0 \end{pmatrix}$ $a_2 = \begin{pmatrix} 0 \\ -id_V \\ 0 \end{pmatrix}$ and $b_0 = (-y\ x\ i)$, $b_1 = (0 - id_V\ 0)$, $b_2 = (id_V\ 0\ 0)$.

The monad can now be put in the more convenient form

$$V \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow[a=\begin{pmatrix} z_0x - z_1 \\ z_0y - z_2 \\ z_0j \end{pmatrix}]{} \begin{matrix} V \otimes \mathcal{O}_{\mathbb{P}^2} \\ \oplus \\ V \otimes \mathcal{O}_{\mathbb{P}^2} \\ \oplus \\ W \otimes \mathcal{O}_{\mathbb{P}^2} \end{matrix} \xrightarrow[b=(-(z_0y-z_2)\ z_0x-z_1\ z_0i)]{} V \otimes \mathcal{O}_{\mathbb{P}^2}(1)$$

To establish the isomorphism of our moduli space of sheaves on $\mathbb{P}^2$ with Nakajima quiver variety, it remains to prove the following lemma.

Lemma. Suppose the quadruple (x, y, i, j) satisfies the equation $[x, y] + ij = 0$. For a and b constructed as above

$(1) \ker a = 0$

$(2) b$ is surjective if and only if the stability condition holds, namely, there is no $S \subset \mathbb{C}^n$, such that $x(S), y(S) \subset S$ and $im(i) \subset S$.

Proof. It follows from the discussion above, that a is injective and b surjective on ℓ_∞. To prove (1), notice that if there is a $v \in V$, such that $v \in \ker a$ for a point $(z_1, z_2) \in \mathbb{C}^2 = \mathbb{P}^2 \backslash \ell_\infty$, then

$$\begin{cases} xv = z_1 v \\ yv = z_2 v \\ z_2 jv = 0, \end{cases}$$

which can clearly happen only for a finite number of points (z_1, z_2) and, therefore, a is injective, when restricted to any open neighborhood of any point in $\mathbb{C}^2$.

Suppose b is surjective, but there exists $S \subset \mathbb{C}^n$, contradicting the assertion. We look at the dual operators x^t, y^t, i^t, j^t acting on $\mathbb{C}^{n*}$ and $\mathbb{C}^{r*}$, and introduce $S^\perp := \{\phi \in \mathbb{C}^{n*} | \phi(S) = 0\}$. The condition $im(i) \subset S$ is equivalent to $S^\perp \subset \ker i^t$. It is not hard see that the equation $[x, y] + ij = 0$ induces $[x^t, y^t] + j^t i^t = 0$. Thus it follows that x^t and y^t commute on $S^\perp$ (it is preserved by x^t and y^t, because $x(S), y(S) \subset S$) and, therefore have a common eigenvector φ with eigenvalues $(\lambda_1, \lambda_2) \in \mathbb{C}$, so b^t is not injective at the point (λ_1, λ_2), dually, b is not surjective at some point, hence, not surjective.

To prove the converse, just reverse the above argument and take $S = \ker \varphi$.

□

3. Torus Action on $\mathcal{M}_{r,n}$

3.1. **Torus Action on Hilbert Scheme of Points.** Let us remind that for the Hilbert scheme of n points on $\mathbb{C}^2$, which consists of ideals $I \subset \mathbb{C}[x,y]$ of codimension n, the 2-dimensional torus action comes from the action on $\mathbb{C}^2$, defined by $(t_1, t_2) \in (\mathbb{C}^*)^2 : (z_1, z_2) \mapsto (t_1 z_1, t_2 z_2)$. Thus, the only invariant point is $0 \in \mathbb{C}^2$ and invariant points of the Hilbert scheme are ideals supported on 0. It is not hard to see that such ideals are generated by monomials. It is convenient to encode them with Young diagrams.

3.2. **Fixed Points Set for Torus Action on $\mathcal{M}_{r,n}$.** To find the fixed points set for the torus $T \times (\mathbb{C}^*)^2$ (T is maximal torus in $GL(W)$) action on $\mathcal{M}_{r,n}$, we decompose $W = W_1 \oplus W_2 \oplus \cdots \oplus W_r$ as the sum of weight spaces with respect to T-action. The torus fixed points are then $(\mathcal{M}_{1,n_1})^{(\mathbb{C}^*)^2} \times \cdots \times (\mathcal{M}_{1,n_r})^{(\mathbb{C}^*)^2}$, $\sum_{i=1}^{r} n_i = n$, and can be encoded via multipartitions.

4. Appendix A

Theorem 1. Let G, F be coherent sheaves on a compact variety X, moreover, F is locally free. Then $Rpr_{1*}(F \boxtimes G) \cong F \otimes H^\bullet(G)$.

Proof. Choose a Cech resolution $C^\bullet$ of G, it will be of finite length, because X is compact. Using that G is quasi isomorphic to $C^\bullet$ in $D^b(X)$, the functors $\otimes F$ and pr_2^* are exact, we get that $F \boxtimes G \cong F \boxtimes C^\bullet$ in $D^b(X \times X)$, thus, $Rpr_{1*}(F \boxtimes G) \cong Rpr_{1*}(F \boxtimes C^\bullet) \overset{(1)}{\cong} F \otimes H^0(C^\bullet) \cong F \otimes H^\bullet(G)$, where (1) follows from the projection formula. □

Theorem 2. Let E be a torsion-free coherent sheaf on $\mathbb{P}^2$, locally free on ℓ_∞, then

$$\begin{cases} H^q(\mathbb{P}^2, E(-p)) = 0, & p = 1,2, q = 0,2 \\ H^q(\mathbb{P}^2, E(-1) \otimes Q^\vee) = 0, & q = 0,2 \end{cases}.$$

Proof. Introduce coordinates $[z_0 : z_1 : z_2]$ on $\mathbb{P}^2$ and consider the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{z_0} \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_{\ell_\infty} \to 0,$$

tensor it with $E(-k)$ to come up with

$$0 \to E(-k-1) \to E(-k) \to E(-k)|_{l_\infty} \to 0.$$

This gives the long exact sequence

$$0 \to H^0(\mathbb{P}^2, E(-k-1)) \to H^0(\mathbb{P}^2, E(-k)) \to H^0(\ell_\infty, E(-k)|_{l_\infty})$$

$$\to H^1(\mathbb{P}^2, E(-k-1)) \to H^1(\mathbb{P}^2, E(-k)) \to H^1(\ell_\infty, E(-k)|_{\ell_\infty}) \\ \to H^2(\mathbb{P}^2, E(-k-1)) \to H^2(\mathbb{P}^2, E(-k)) \to 0.$$

As $E|_{\ell_\infty} \cong \mathcal{O}^{\oplus r}$, we get

$$\begin{cases} H^0(\ell_\infty, E(-k)|_{\ell_\infty}) = 0, & k \geqslant 1 \\ H^1(\ell_\infty, E(-k)|_{\ell_\infty}) = 0, & k \leqslant 1 \end{cases}$$

Thus from the exact sequence we see that

$$\begin{cases} H^0(\mathbb{P}^2, E(-k-1)) \cong H^0(\mathbb{P}^2, E(-k)), & k \geqslant 1 \\ H^2(\mathbb{P}^2, E(-k-1)) \cong H^2(\mathbb{P}^2, E(-k)), & k \leqslant 1 \end{cases}$$

By Serre vanishing theorem $H^2(\mathbb{P}^2, E(n)) = 0$ for $n \in \mathbb{N}$ large enough, while duality asserts that $H^0(\mathbb{P}^2, E(-n)) \cong H^2(\mathbb{P}^2, E^\vee(n) \otimes K_{\mathbb{P}_2}) \cong H^2(\mathbb{P}^2, E^\vee(n-3)) \cong 0$.

$$\begin{cases} H^0(\mathbb{P}^2, E(-1)) \cong H^0(\mathbb{P}^2, E(-2)) \cong \cdots = 0 \\ H^2(\mathbb{P}^2, E(-2)) \cong H^2(\mathbb{P}^2, E(-1)) \cong \cdots = 0. \end{cases}$$

The proof of the second assertion of the theorem is similar (see [Nak99]): consider the sequence

$$0 \to E(-k-1) \otimes Q^\vee \to E(-k) \otimes Q^\vee \to (E(-k) \otimes Q^\vee)|_{l_\infty} \to 0, \\ Q|_{\ell_\infty} \cong \mathcal{O}|_{\ell_\infty} \oplus \mathcal{O}|_{\ell_\infty}(1).$$

□

References

[Nak99] H. Nakajima, *Lectures on Hilbert schemes of points on surfaces*, University Lecture Series, vol. 18, American Mathematical Society, Providence, RI, 1999.

[OSS11] C. Okonek, M. Schneider, and H. Spindler, *Vector bundles on complex projective spaces*, Modern Birkhäuser Classics, Birkhäuser/Springer Basel AG, Basel, 2011.

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